McDuff superrigidity for group II_1 factors

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Group von Neumann algebras

Throughout, G will denote a countable discrete group.

- Left regular representation: $u : G \to \mathcal{U}(\ell^2 G)$
  Where $(u_g \xi)(h) = \xi(g^{-1}h)$ for any $g, h \in G$ and $\xi \in \ell^2 G$.
- Induces an injective $*$-homomorphism $\mathbb{C}[G] \hookrightarrow \mathcal{B}(\ell^2 G)$.

The **group von Neumann algebra** is given by $\mathcal{L}G = \overline{\mathbb{C}[G]}^{sot} \subset \mathcal{B}(\ell^2 G)$.

- $\mathcal{L}G$ is a tracial von Neumann algebra, with the canonical faithful normal tracial state τ extending $\tau(u_g) = \delta_{e,g}$.

Theorem (Murray-von Neumann '43)

*$\mathcal{L}G$ is a II_1 **factor** if and only if G is **ICC** (infinite conjugacy classes).*

Rigidity for group von Neumann algebras

Question: How much does $\mathcal{L}G$ remember from G?
When and what properties of G pass to $\mathcal{L}G$?

- **Locally finite ICC groups** G have $\mathcal{L}G \cong \mathcal{R}$ (Murray-von Neumann '43).
- If G is **ICC amenable**, then $\mathcal{L}G \cong \mathcal{R}$ (Connes '76).
- $\mathcal{L}\mathbb{F}_n$ is **prime** for $n \geq 2$ (Ge '96). Same for any **ICC hyperbolic** group (Ozawa '03).
- **Product rigidity** $\mathcal{L}(G_1 \times ... \times G_n)^t = \mathcal{L}H$, then $H = H_1 \times ... \times H_n$ with $\mathcal{L}H_i = \mathcal{L}G_i$ (up to amplifications and an inner automorphism) for $G_i \in \mathcal{S}_{nf}$ weakly amenable (Chifan-de Santiago-Sinclair '15).
- First examples of ***W**-superrigid groups** ($\mathcal{L}G = \mathcal{L}H$ implies $G \cong H$) constructed from wreath products (Ioana-Popa-Vaes '10).

The hyperfinite II_1 factor $\mathcal{R}$ and the McDuff property

The hyperfinite II_1 factor $\mathcal{R}$

- $\mathcal{R} \cong \overline{\bigotimes}_{\mathbb{N}}(M_2(\mathbb{C}), \tau) \cong \mathcal{R}\overline{\otimes}\mathcal{R}$.

McDuff Property

A factor $\mathcal{M}$ is said to be **McDuff** if $\mathcal{M} \cong \mathcal{M}\overline{\otimes}\mathcal{R}$.

- If $\mathcal{L}G$ is McDuff, then $\mathcal{L}G \cong \mathcal{L}G\overline{\otimes}\mathcal{L}A \cong \mathcal{L}(G \times A)$ (for any A amenable ICC).
- We call a group ***W**-McDuff** if $\mathcal{L}G$ is McDuff.

Absence of rigidity: W^*-McDuff groups can never be W^*-superrigid.

McDuff W^*-superrigidity

Question: Can we have instances of non W^*-superrigid groups where the lack of rigidity is completely characterized?

Any rigidity statement on W^*-McDuff groups has to allow for a direct sum with an ICC amenable group.

Definition

We say a group G is **McDuff superrigid** if for any discrete group H we have that $\mathcal{L}G = \mathcal{L}H$ implies $H \cong G \times A$ (for some ICC amenable group A).

i. e. for McDuff superrigid groups, the obvious obstruction to superrigidity is the only obstruction.

Infinite product rigidity

Infinite products (direct sums):

- Take a collection $\{G_n\}_{n\in\mathbb{N}}$ of infinite ICC groups.
- Then, $G = \bigoplus_{\mathbb{N}} G_n$ is W^*-McDuff, since $\mathcal{L}G \cong \overline{\bigotimes}_{\mathbb{N}} \mathcal{L}G_n$ and

$$\overline{\bigotimes}_{\mathbb{N}} \mathcal{L}G_n \cong \overline{\bigotimes}_{\mathbb{N}} (p_n(\mathcal{L}G_n)p_n \otimes M_2(\mathbb{C})) \cong \overline{\bigotimes}_{\mathbb{N}} (p_n(\mathcal{L}G_n)p_n)\overline{\otimes}\mathcal{R}\,.$$

- Any ICC infinite product group is W^*-McDuff.

Theorem (Infinite product rigidity, Chifan-Udrea '18)

Let $\{G_n\}_{n\in\mathbb{N}}$ be an infinite collection of infinite ICC, weakly amenable, bi-exact, property (T) groups and $G = \bigoplus_{\mathbb{N}} G_n$. Suppose $\mathcal{L}G = \mathcal{L}H$ for some discrete group H. Then $H = \bigoplus_{\mathbb{N}} H_n \times A$ with $\mathcal{L}(G_n) \cong \mathcal{L}(H_n)$ (up to an amplification) and A is an amenable ICC group.

Approach to finding McDuff W^*-superrigid groups

If only we could plug in **W^*-superrigid** groups G_n (for $n \in \mathbb{N}$) into the **Infinite Product Rigidity** of (Chifan-Udrea '18),

then for $G = \bigoplus_{\mathbb{N}} G_n$ and $\mathcal{L}G = \mathcal{L}H$ we would get:

$H = \bigoplus_{\mathbb{N}} H_n \times A$ with $\mathcal{L}(G_n) \cong \mathcal{L}(H_n)$ and A is some amenable ICC group.

Thus, G_n being W^*-superrigid would imply $G_n \cong H_n$ and therefore $H \cong \bigoplus_{\mathbb{N}} G_n \times A = G \times A$.

Obstacles to directly using existing results

- The infinite product rigidity result of Chifan-Udrea '18 requires each group in the direct sum to be weakly amenable, bi-exact and have property (T).
- To deduce McDuff W^*-superrigidity we would also need each direct summand to be W^*-superrigid.

Chifan-Ioana-Osin-Sun '21: First examples of W^*-superrigid groups with property (T), through a novel construction called wreath-like products.

- But now, it is not known whether the W^*-superrigid wreath-like product groups could be bi-exact (or weakly amenable).

Wreath-like product groups

Wreath-like product: Take groups A, B and an action $B \curvearrowright I$ on a set.

Then $G \in \mathcal{WR}(A, B \curvearrowright I)$, if there is a s.e.s.

$$1 \longrightarrow A^{(I)} := \bigoplus_I A_i \longrightarrow G \xrightarrow{\varepsilon} B \longrightarrow 1$$

where $A_i \cong A$ and such that $gA_ig^{-1} = A_{\varepsilon(g)\cdot i}$ for any $i \in I$, $g \in G$.

Wreath-like product groups as cocycle semidirect products

Wreath-like product: Take groups A, B and an action $B \curvearrowright I$ on a set.

Then $G \in \mathcal{WR}(A, B \curvearrowright I)$, if there is a s.e.s.

$$1 \longrightarrow A^{(I)} := \bigoplus_I A_i \longrightarrow G \underset{\zeta}{\overset{\varepsilon}{\rightleftarrows}} B \longrightarrow 1$$

where $A_i \cong A$ and such that $gA_ig^{-1} = A_{\varepsilon(g)\cdot i}$ for any $i \in I$, $g \in G$.

Take any set-theoretic section $\zeta : B \to G$.

Then we have a map $\sigma : B \to \text{Aut}(A^{(I)})$ given by $\sigma(b) = \text{Ad}(\zeta(b))$

and a 2-cocycle $\alpha : B \times B \to A^{(I)}$ given by $\alpha(b, b') = \zeta(b)\zeta(b')\zeta(bb')^{-1}$.

Thus, $G \cong A^{(I)} \rtimes_{\sigma,\alpha} B$ is a cocycle-twisted semidirect product.

Similarly, $\mathcal{L}(G) = \mathcal{L}(A^{(I)}) \rtimes_{\sigma,\alpha} B$ is a cocycle crossed product.

The main issue for Inf. Prod. Rig. with $\mathcal{WR}(A, B \curvearrowright I)$

To compare the group structures in $\mathcal{L}G$ and $\mathcal{L}H$ we look through the comultiplication $\Delta : \mathcal{L}H \to \mathcal{L}H \bar{\otimes} \mathcal{L}H$ which maps the canonical group unitaries by $u_h \mapsto u_h \otimes u_h$ for each $h \in H$.

We want to be able to locate tails of the infinite product, i. e. we want for each $i \in \mathbb{N}$ some $j \in \mathbb{N}$ such that

$$\Delta(\mathcal{L}(\bigoplus_{\mathbb{N}\setminus\{i\}} G_n)) \prec \mathcal{L}(\bigoplus_{\mathbb{N}\setminus\{j\}} G_n) \bar{\otimes} \mathcal{L}(\bigoplus_{\mathbb{N}\setminus\{j\}} G_n)$$

We do have that G_n is bi-exact relative to $A_n^{(I_n)}$, and this leads to

$$\Delta(\mathcal{L}(\bigoplus_{\mathbb{N}\setminus\{i\}} G_n)) \prec \mathcal{L}(\bigoplus_{\mathbb{N}\setminus\{j\}} G_n \times A_j^{(I_j)}) \bar{\otimes} \mathcal{L}(\bigoplus_{\mathbb{N}\setminus\{j\}} G_n \times A_j^{(I_j)})$$

The main issue for Inf. Prod. Rig. with $\mathcal{WR}(A, B \curvearrowright I)$

We need to bypass $\mathcal{L}(A_j^{(I_j)}) = \overline{\bigotimes}_{I_j}(A_j)$ in the intertwining

$$\Delta(\mathcal{L}(\bigoplus_{\mathbb{N}\setminus\{i\}} G_n)) \prec \mathcal{L}(\bigoplus_{\mathbb{N}\setminus\{j\}} G_n \times A_j^{(I_j)}) \bar{\otimes} \mathcal{L}(\bigoplus_{\mathbb{N}\setminus\{j\}} G_n \times A_j^{(I_j)})$$

From Ioana '06 there is a deformation of $\mathcal{L}(A_j^{(I_j)}) = \overline{\bigotimes}_{I_j}\mathcal{L}A_j$ that controls how far into an infinite tensor product one can intertwine.

More specifically, there is a dilation $\mathcal{L}(A_j^{(I_j)}) \subset \widetilde{M} := \overline{\bigotimes}_{I_j}\mathcal{L}(A_j * \mathbb{Z})$ and a path of automorphisms $\alpha_t \in \text{Aut}(\widetilde{M})$ that "move" each A_j towards $\mathbb{Z}$.

By controlling the rate of convergence of $\alpha_t \xrightarrow{t\to 0} id$ we can have a finite $K \Subset I_j$ such that

$$\Delta(\mathcal{L}(\bigoplus_{\mathbb{N}\setminus\{i\}} G_n)) \prec \mathcal{L}(\bigoplus_{\mathbb{N}\setminus\{j\}} G_n) \bar{\otimes} (\overline{\bigotimes}_K (A_j)) \bar{\otimes} \mathcal{L}(\bigoplus_{\mathbb{N}\setminus\{j\}} G_n) \bar{\otimes} (\overline{\bigotimes}_K (A_j))$$

Arrays for Wreath-like product groups in $\mathcal{WR}_b(A, B \curvearrowright I)$

Consider $\mathcal{H}_{\mathbb{R}} = \ell^2_{\mathbb{R}}(I)$ with the action of $G = A^{(I)} \rtimes_{\sigma,\alpha} B$ (wreath-like product group), given by $\pi : G \to B \to \mathcal{O}(\ell^2(I)_{\mathbb{R}})$

and a map $q : G \to \ell^2(I)_{\mathbb{R}}$ which reads which entries of $(a_i)_i \in A^{(I)}$ are different from the identity, i. e. for $g = cb$ with $c \in A^{(I)}, b \in B$ we let

$$q(cb)(i) = \begin{cases} 1 & \text{if } i \in \operatorname{supp}(c) \\ 0 & \text{otherwise.} \end{cases}$$

We guarantee q is an **array** (a generalization of a 1-cocycle) by assuming the 2-cocycle $\alpha : B \times B \to A^{(I)}$ has uniformly bounded support (and then say $G \in \mathcal{WR}_b(A, B \curvearrowright I)$).

This means, there is $D_0 > 0$ such that $|\operatorname{supp}(\alpha(b, b'))| < D_0$ for all $b, b' \in B$ where $\operatorname{supp}(\alpha(b, b')) = \{i \in I : \alpha(b, b')_i \neq 1\}$.

Subordinating two deformations on $\mathcal{WR}_b(A, B \curvearrowright I)$

Using the array q on G we can apply the Gaussian deformation to $\mathcal{L}(G_j)$ from (Sinclair '10, Chifan-Sinclair '11).

We also have the tensor length deformation α_t on $\mathcal{L}(A_j^{(I_j)})$.

We can tie together the rate of convergence of the two deformations on subalgebras of $\mathcal{L}(A_j^{(I_j)})$.

Moreover, for $\{G_n \in \mathcal{WR}_b(A_n, B_n \curvearrowright I_n) : n \in \mathbb{N}\}$, the support arrays q_n on G_n as before can be tensored together to form an array on $\bigoplus_{\mathbb{N}} G_n$ that still captures information about the tensor length in finitely many of the $A_n^{(I_n)}$.

(An **array** for $\pi : G \to \mathcal{O}(\mathcal{H}_{\mathbb{R}})$ is a map $q : G \to \mathcal{H}_{\mathbb{R}}$ such that for all $g \in G$ there is $0 < K < \infty$ so $\sup_{h \in G} \|q(gh) - \pi_g q(h)\| < K$. Tensoring two arrays generally increases the value of K)

Constructing groups in $\mathcal{WR}_b(A, B \curvearrowright I)$

Theorem (AM-Chifan-Osin-Sun '25)

Let G be a torsion-free hyperbolic group. Suppose that g is a non-trivial element of G that is not a proper power. Then for any sufficiently large prime $n \in \mathbb{N}$, we have

$$G/[\langle\langle g^n \rangle\rangle, \langle\langle g^n \rangle\rangle] \in \mathcal{WR}_b(\mathbb{Z}, G/\langle\langle g^n \rangle\rangle \curvearrowright I),$$

where the action of $G/\langle\langle g^n \rangle\rangle$ on I is transitive with stabilizers isomorphic to $\mathbb{Z}/n\mathbb{Z}$.
In addition, $G/\langle\langle g^n \rangle\rangle$ is a non-elementary, ICC, hyperbolic group.

Infinite product rigidity for $\mathcal{WR}_b(A, B \curvearrowright I)$

Theorem (AM-Chifan-Osin-Sun '25)

For $n \in \mathbb{N}$ let $G_n \in \mathcal{WR}_b(A_n, B_n \curvearrowright I_n)$ be a property (T) wreath-like product group where A_n is a nontrivial amenable group, B_n is an ICC subgroup of a hyperbolic group and the action $B_n \curvearrowright I_n$ has amenable stabilizers. Denote $G = \oplus_{n\in\mathbb{N}} G_n$ and assume that H is an arbitrary group satisfying $\mathcal{L}(G) = \mathcal{L}(H)$.
Then H admits an infinite direct sum decomposition $H = (\oplus_{n\in\mathbb{N}} H_n) \times A$, with $\mathcal{L}(G_n) \cong \mathcal{L}(H_n)$ (up to an amplification) for some A ICC amenable group.

McDuff superrigid groups

From Chifan-Ioana-Osin-Sun '21 we know many groups G_n that fit into the previous theorem are actually W^*-superrigid (and have trivial fundamental group).

Theorem (AM-Chifan-Osin-Sun '25)

For every $n \in \mathbb{N}$ let $G_n \in \mathcal{WR}_b(A_n, B_n \curvearrowright I_n)$ be a property (T) wreath-like product group where A_n is a nontrivial abelian group, B_n is an ICC subgroup of a hyperbolic group and the action $B_n \curvearrowright I_n$ has amenable stabilizers. Denote by $G = \oplus_{n\in\mathbb{N}} G_n$.

Then G is McDuff superrigid.

Corollary (AM-Chifan-Osin-Sun '25)

There exists a continuum of pairwise non-isomorphic McDuff superrigid groups.

Thank you!